The Influence of Acceleration on Laminar Similar Boundary Layers

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Strongly accelerating, laminar compressible boundary layers are considered. It is argued that similar solutions are a good approximation of the nonsimilar case. The mathematical properties of the solutions are discussed and the bounds for the skin-friction and heat-transfer coefficients are found. Convenient and reliable correlations for both coefficients, as well as the velocity and temperature profiles, are suggested.

Introduction

Background

THIS paper is concerned with the similar solution of compressible, two-dimensional boundary-layer equations. Compressible boundary layers form a very important group of flows, the efficient solution of which is of great importance to aerodynamicists. The general case of nonsimilar boundary layers requires the solution of a parabolic system of equations, which is both difficult and expensive. However, similar boundary layers are governed by a set of two coupled, ordinary differential equations that may be solved relatively inexpensively. In general, the nonsimilar solution depends on the history of the flow along the integration path. However, it may be argued that, if the variations of the pressure gradient in the streamwise direction are minor, the difference between the similar and nonsimilar solutions is not very large. This point, which is borne out by our experience, is demonstrated in Fig. 1 where a nonsimilar solution is compared to a similar solution for the same pressure gradient and wall temperature. Very clearly, the two velocity profiles are not very different. Typically, the skin friction and heat coefficient for the two calculations differ from one another by only 5-7%. Another useful assumption is that of a model fluid ($Pr=1$ and $\mu\alpha T$) that causes relatively small errors when the airflow is considered.

Literature Survey

Thus, it appears that the similar solution of the boundary-layer equation for a model fluid may form a very useful approximation of the nonsimilar boundary layers. This paper is concerned with such solutions—their properties and their reasonable approximations. As the two coupled, nonlinear ordinary differential equations for the momentum and enthalpy are not amenable to analytical treatment, numerical methods are usually required. However, even numerical solutions are difficult due to the severe velocity gradient near the wall. Therefore, numerical solutions have been obtained only up to a relatively low-pressure gradient parameter $\beta=20$ by, e.g., Christian et al.,[1] Back,[2] and Back and Cuffel.[3] These solutions are given in tables; but Forbrich,[4] who solved the equations for $\beta\leq 2$, suggested an empirical relation between the skin friction f_w'' and the pressure gradient β for the adiabatic case. Another interesting point is the convergence of the solutions for large β toward an asymptotic solution. We believe that the asymptotic solution is a reasonably good approximation for $\beta\geq 50$. However, the reported limiting solution[5] does not represent a good approximation for high-pressure gradient flows, as will be demonstrated below.

The problem of existence and uniqueness deserves some attention, not only because we wish to confirm the good behavior of the system, but also because their study allows us to find the bounds of various quantities. However, the mathematical properties of this set of equations were not investigated and the research was restricted to the properties of the uncoupled momentum equation (corresponding to an adiabatic wall). This was investigated by Coppel[6] and followed later by others (e.g., Veldman and Van der Vooren[7]).

Purpose of the Paper

The aim of this research was to study the system of equations governing the self-similar case. The first and most important goal was to formulate simple approximate expressions for the important quantities like skin friction, wall heat flux, and velocity and temperature profiles. This goal was obtained by a combination of exact analysis and empirical correlations of the numerical solutions. We shall also present uniqueness and existence theorems and discuss the problem of a limiting solution for infinite β. The paper is restricted to accelerating laminar compressible flow of nonreacting model fluid over isothermal-cooled walls.

Mathematical Formulation

The Governing Equations

The governing equations for steady, laminar, compressible boundary layers are

$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0 \tag{1}$$

$$\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{\partial(\mu\partial u/\partial y)}{\partial y} \tag{2}$$

$$\rho u\frac{\partial h_s}{\partial x}+\rho v\frac{\partial h_s}{\partial y}=\frac{\partial(\mu/Pr\partial h_s/\partial y)}{\partial y}+\frac{\partial[\mu(1-1/Pr)\partial u^2/2)/\partial y]}{\partial y} \tag{3}$$

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where x and y are the Cartesian coordinates parallel and normal to the flow direction, respectively; ρ, p, u, and v the density, pressure, and velocity components in the x and y directions, respectively; h_s the local stagnation enthalpy; μ the viscosity; and Pr the Prandtl number. The discussion is limited to model fluid, i.e., to the case in which $Pr=1$ and $\mu=\mu_r(T/T_r)$, where T denotes the temperature and μ_r and T_r the reference values of μ and T. Assuming constant specific heats using the Illingworth-Stewartson or Levy Lees transformation, the following nonlinear, coupled, ordinary set of equations is obtained for the similar boundary layer:

$$f''' + ff'' + \beta(g - f'^2) = 0 \tag{4a}$$

$$g'' + fg' = 0 \tag{4b}$$

and

$$\xi = \int \rho_e u_e \mu_e \mathrm{d}x$$

$$\eta = \frac{\int(\rho/\rho_e)\mathrm{d}y}{\delta(x)}$$

$$\delta(x) = (2\xi)^{1/2}/\rho_e u_e \tag{5}$$

$$\beta = \frac{(T_{s,e}/T_e)2\xi}{u_e}\frac{\mathrm{d}u_e}{\mathrm{d}\xi} = \frac{2\xi}{M_e}\frac{\mathrm{d}M_e}{\mathrm{d}\xi} \tag{6}$$

where T_e denotes the local stagnation temperature, M_e the Mach number, and subscript e the values of the corresponding variables at the edge of the boundary layer. The functions f and g are defined by

$$f'(\eta) = u/u_e \tag{7}$$

$$g(\eta) = \frac{T + u^2/2C_p}{T_{s,e}} = \frac{T_s}{T_{s,e}} \tag{8}$$

where C_p is the specific heat capacity of the fluid at constant pressure.

The boundary conditions for Eqs. (4) are

$$f(0)=0, \quad f'(0)=0, \quad \lim_{\eta\to\infty} f'(\eta)=1 \tag{9a}$$

$$g(0)=g_w, \quad \lim_{\eta\to\infty} g(\eta)=1 \tag{9b}$$

It will be noted that this study is concerned with isothermal walls only. However, the adiabatic wall case is included when $g_w=1$. By definition, g_w is always positive and we shall not discuss the case of heated walls ($g_w>1$), which is of little engineering significance. The pressure gradient parameter β will be free to vary between zero and infinity, but not to have negative values. Thus, this investigation is limited to accelerating flows on cooled walls.

The Numerical Method

Although Eqs. (4) with the boundary conditions [Eqs. (9)] are not amenable to analytical solutions, we solved them numerically using a fourth-order finite difference method. The method has been described elsewhere[8,9] and will be only briefly outlined here. The method is a compact hermitian scheme, where block-tridiagonal systems of equations for the corresponding variable and its derivative are solved. Two such blocks are solved, for the velocity $f'(\eta)$ and the stagnation enthalpy $g(\eta)$. The stream function $f(\eta)$ is obtained by fourth-order generalized Simpson integration of $f'(\eta)$. The following exponential coordinate transformation was applied:

$$\zeta = 1 - \exp(-A\eta) \tag{10}$$

In order to minimize the number of mesh points, typically we choose $A=3\div5$. With this stretching, about eight mesh points are sufficient to obtain a good solution.

Properties of the Solutions

Coppel[6] studied the uniqueness and existence of the solution of Eq. (4a) for a favorable pressure gradient ($\beta>0$) subject to the boundary condition of Eq. (9a) and with $g_w=1$ corresponding to an adiabatic wall. He proved the existence of a unique solution that satisfies the following conditions:

$$0 \le f'(\eta) \le 1 \tag{11a}$$

$$0 \le f''(0) \tag{11b}$$

He also showed that

$$\left(\frac{4\beta}{3}\right)^{1/2} \le f''_w \le \left(\frac{4\beta+1}{3}\right)^{1/2} \tag{12}$$

Equation (12) readily suggests that for the adiabatic case $g_w=1$

$$\lim_{\beta\to\infty} \frac{f''_w}{(4\beta/3)^{1/2}} = 1 \tag{13}$$

Coppel's condition[6] may be extended[10] to the nonadiabatic case to give

$$\left(\frac{4g_w\beta}{3}\right)^{1/2} \le f''_w \le \left(\frac{4\beta+1}{3}\right)^{1/2} \tag{14}$$

Therefore, the following condition is satisfied for any positive β:

$$f''_w = \phi(g_w)(4\beta/3)^{1/2} \tag{15}$$

where ϕ becomes infinite for $\beta=0$, but is finite for positive β, and of order unity for $\beta\to\infty$. Pade et al.[10] show that in the nonadiabatic case a unique solution exists which satisfies conditions of Eqs. (11) as well as the following two additional conditions:

$$0 \le g(\eta) \le 1 \tag{16a}$$

$$g(\eta) - f'^2(\eta) \ge 0 \tag{16b}$$

Equation (16a) is analogous to Eq. (1a). Equation (16b) is clearly valid for $\eta=0$ and $\eta=\infty$. We argue that Eq. (16b) makes sense due to the squaring of f' on the left-hand side of the inequality. This condition is indeed manifested by all our numerical calculations as shown in Fig. 2 for the specific case of $g_w=0.2$.

Pade et al.[10] show that the solution of Eqs. (4) is monotonous with β, that is to say,

$$[f'(\eta)]_{\beta_1} \ge [f'(\eta)]_{\beta_2} \tag{17a}$$

$$[g(\eta)]_{\beta_1} \ge [g(\eta)]_{\beta_2} \tag{17b}$$

for $\beta_1 \ge \beta_2$.

Some Analytical Considerations

Formal expressions for f''_w and g'_w may be obtained from Eqs. (4) with the boundary conditions of Eqs. (9),

$$h(\eta) = f''_w - \beta\int\exp\left(\int f\mathrm{d}\eta\right)(g-f'^2)\mathrm{d}\eta$$

$$f''(\eta) = h(\eta)\exp\left(-\int f\mathrm{d}\eta\right) \tag{18}$$

$$g'(\eta) = g'_w\exp\left(-\int f\mathrm{d}\eta\right) \tag{19}$$

$$g(\eta) = g_w + g'_w\int\exp\left(-\int f\mathrm{d}\eta\right)\mathrm{d}\eta \tag{20}$$

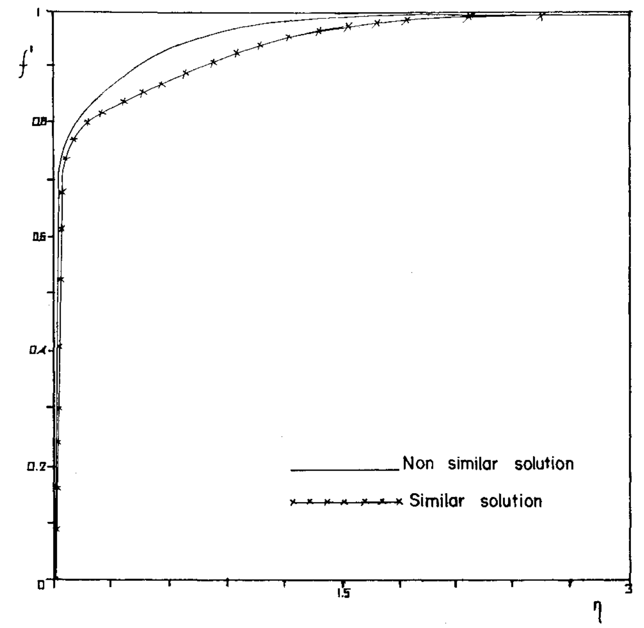

Fig. 1 Comparison of similar and nonsimilar solutions ($g_w = 0.6$, $M_e = 1.8$, $\beta = 1000$).

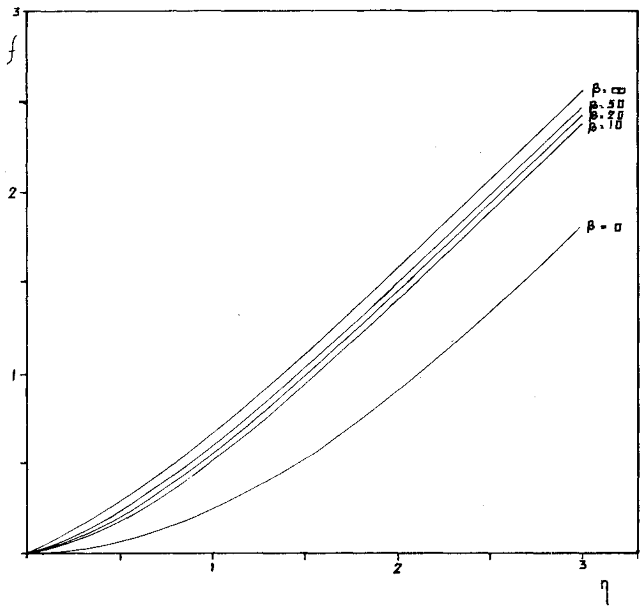

Fig. 3 Dimensionless stream function f for various β ($g_w = 0.2$).

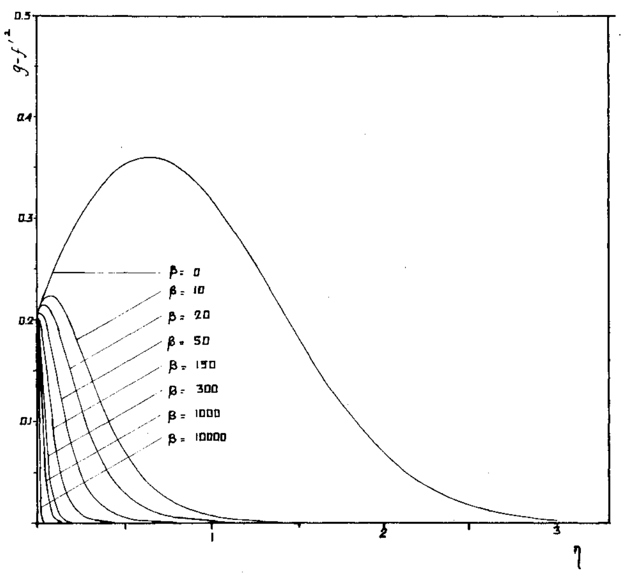

Fig. 2 Function of $g - f'^2$ for various β ($g_w = 0.2$).

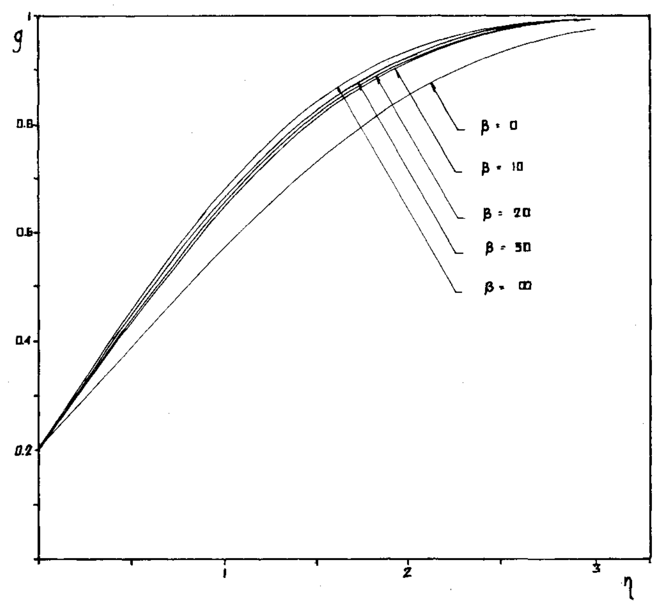

Fig. 4 Dimensionless stagnation temperature g for various β ($g_w = 0.2$).

On the wall

$$g'_w = (1 - g_w)/I$$

where $I = \int[\exp(-\int f d\eta)] d\eta \geq (\pi/2)^{1/2}$. Therefore,

$$g'_w \leq (1 - g_w)(2/\pi)^{1/2} \tag{21}$$

As $f''(\eta) \geq 0$, it follows from Eq. (18) that

$$\beta \int \exp(\int f d\eta)(g - f'^2) d\eta \leq f''_w$$

And using the right-hand side of Eq. (14), we may write

$$\int \exp(\int f d\eta)(g - f'^2) d\eta \leq \left[\frac{4 + (1/\beta)}{3\beta}\right]^{1/2}$$

Therefore,

$$\lim_{\beta \to \infty} (g - f'^2) = 0 \tag{22}$$

Solutions for Very Large Pressure Gradients

Figures 3 and 4 show the velocity f' and enthalpy g profiles for various pressure gradients β. When β increases, the solutions appear to converge toward an asymptotic solution. It would have been useful to find an analytical expression for this solution that could be used as a good approximation for cases of very-high-pressure gradient parameters. One way to achieve this goal is by solving the asymptotic form of the equations. Evans[5] found this asymptotic form and obtained a solution for the adiabatic case $g_w = 1$. However, comparison with the present solution shows that his solution is not the limit of the solutions of Eq. (4). This problem may be understood by noting that Eq. (22) cannot be satisfied on the wall. Therefore, $f'(\eta)$ is not continuous for $\beta \to \infty$ at $\eta = 0$. This singularity makes it impossible to obtain the limiting solution from Eq. (4a), but as Eq. (4b) does not contain the discontinuous f', Pade et al.[10] show that a unique limiting solution of $f(\eta)$ and $g(\eta)$ for $\beta \to \infty$ exists. We may note that $f(\eta)$ is continuous but not differentiable at $\eta = 0$. The limiting solution is calculated by substitution of

$$\lim_{\beta\to\infty} f(\eta) = \int \lim_{\beta\to\infty} [g(\eta)^{1/2}] d\eta$$

obtaining the integrodifferential equation,

$$g'' + [\int [g(\eta)^{1/2}] d\eta] g' = 0 \quad (23)$$

Numerical solution for f and g as obtained from Eq. (23) are shown in Figs. 3 and 4 together with the solutions for very large pressure gradient parameters β. The profiles are very similar, thus showing that Eq. (23) is the asymptotic form of Eqs. (4). Moreover, it becomes obvious that, for large pressure gradients (say $\beta > 20$), the asymptotic solution is a very good approximation of the exact solution.

Approximate Distributions

Examination of the properties of both solutions discussed in the previous section and the numerical results suggest that fairly accurate approximations to the solution may be obtained in the form of generalized correlations. Such correlations are very useful, as they relieve us of the need to perform lengthy computations. Indeed, these practices are very common in turbulent flows. In the following paragraphs, we shall present such correlations for the laminar case. The correlations (for wall flux flow and velocity and temperature profiles) are based on the numerical solutions and compare with them favorably.

Wall Flux Laws

Often, we need to know only the skin friction and the wall heat flux, but we do not explicitly need the velocity or temperature profile inside the boundary layer. As the self-similar solutions are very similar to the general solutions (this is particularly so for large pressure gradients), a general correlation for the values of f''_w and g'_w may be useful for engineering calculations. Considering the skin friction, recall that $f''_w = 0.4696$ for zero pressure gradient and [from Eq. (13)] that it approaches the square root of $4\beta/3$ for large pressure gradients and adiabatic walls ($g_w = 1$). With intermediate pressure gradients and nonadiabatic walls, an empirical correlation was obtained by a least square fit as follows:

$$f''_w = 0.4696 + Q(4\beta/3)^{1/2} \quad (24)$$

where

$$Q = 0.09105 + Rg_w - 0.22625 g_w^2 \quad (24a)$$

$$R = (A + B\beta)/(1 + D\beta) \quad (24b)$$

$$A = 0.59467 - 0.2335B \quad (24c)$$

$$D = B/1.1157 \quad (24d)$$

$$B = 2 \quad (24e)$$

The correlation is compared to the exact solution in Fig. 5 and Table 1 for four different values of the wall enthalpy g_w. Good agreement is obtained for most pressure gradient parameters β; the maximum error (at $\beta - 10$) is about 5.8%.

Considering the wall heat flux, we realize that g'_w scales as $(1 - g_w)$ and examination of the scaled results suggests the correlation,

$$g'_w = 0.4696(1 - g_w) + 0.2 g_w^{0.09}(1 - g_w^{1.8})\exp(-\beta^{-0.3}) \quad (25)$$

The wall heat flux as computed from Eq. (25) is compared to the exact solution in Fig. 6 and Table 2. Again, good agreement between the correlations and the exact solution is obtained; the maximum error (at $\beta = 0.5$ and $g_w = 0.8$) is about 6.3%.

The Velocity Profile

An approximate velocity profile is a useful and convenient tool, not only in displaying the influence of parameters β and g_w on the velocity profile, but also for approximating a solution of the boundary-layer equations using integral methods. Such a profile for adiabatic flows were proposed by Pohlhausen[11] for the incompressible case and was later applied by Grushwitz[12] to the compressible case as well. This profile is given by

$$u/u_e = \sigma[2 - \sigma^2(2 - \sigma)] + \Lambda\sigma(1 - \sigma)^3/6$$

where $\sigma = y/\delta$, η_e is the value of η at the outer edge of the boundary layer, and Λ the pressure gradient parameter, which is related to β by $\Lambda = \beta g_w \eta_e^2$ and $0 \le \sigma \le 1$. The dimen-

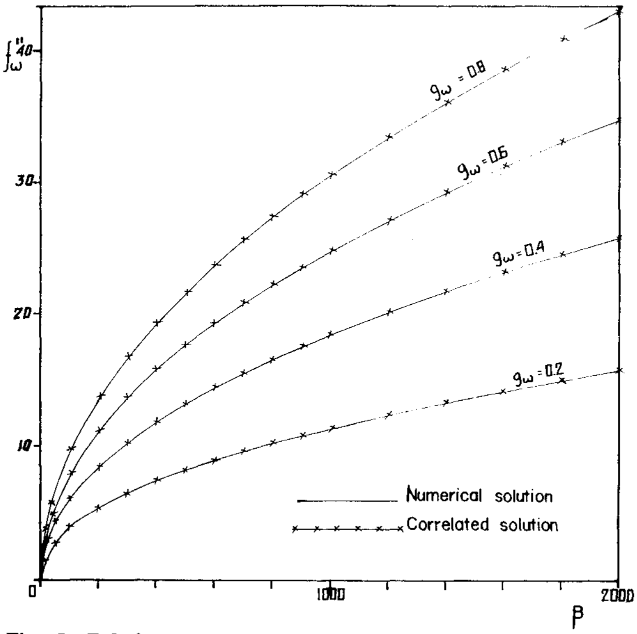

Fig. 5 Relation between skin friction f''_w and pressure gradient parameter β for various wall temperatures g_w.

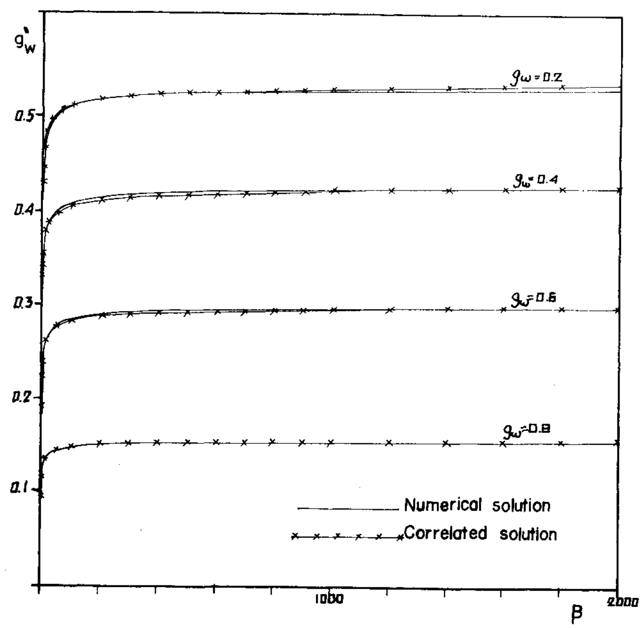

Fig. 6 Relation between wall heat flux g'_w and pressure gradient β for various wall temperature g_w.

Table 1 f_w'' as function of β and g_w

	$g_w = 0.2$		$g_w = 0.4$		$g_w = 0.6$		$g_w = 0.8$	
β	Approx.	Exact	Approx.	Exact	Approx.	Exact	Approx.	Exact
0.0	0.47000	0.47000	0.47000	0.47000	0.47000	0.47000	0.47000	0.47000
0.5	0.65500	0.63406	0.72620	0.71900	0.79520	0.76917	0.86230	0.81455
1.0	0.77600	0.74064	0.89630	0.88524	1.01220	1.00894	1.12400	1.11174
1.5	0.86900	0.82578	1.03100	1.02719	1.18536	1.20301	1.34000	1.35323
2.0	0.94800	0.89792	1.14560	1.14759	1.33340	1.36771	1.51350	1.55827
10.0	1.64200	1.54609	2.19800	2.22362	2.71640	2.83506	3.20700	3.38040
20.0	2.13500	2.01805	2.97600	3.00244	3.75336	3.89337	4.48500	4.69084
50.0	3.08000	2.94366	4.50600	4.52610	5.80700	5.96077	7.02400	7.24764
100.0	4.12600	3.98080	6.22000	6.23123	8.11800	8.27267	9.88600	10.10511
200.0	5.58500	5.44394	8.63580	8.63547	11.38200	11.53142	13.93200	14.13181
300.0	6.69600	6.56546	10.48640	10.47793	13.88500	14.02839	17.03600	17.21686
400.0	7.63000	7.51055	12.04540	12.03039	15.99550	16.13223	19.65300	19.81607
500.0	8.45000	8.34299	13.41850	13.39775	17.85460	17.98516	21.95900	22.10524
600.0	9.19000	9.09547	14.66000	14.63371	19.53530	19.66001	24.04400	24.17436
700.0	9.87500	9.78737	15.80070	15.77016	21.08090	21.19998	25.96200	26.07684
800.0	10.51000	10.43133	16.86300	16.82784	22.52000	22.63321	27.74660	27.84743
900.0	11.10000	11.03612	17.86000	17.82117	23.87100	23.97922	29.42310	29.51027
1000.0	11.66700	11.60811	18.81370	18.76063	25.14900	25.25224	31.00880	31.08292
1200.0	12.71300	12.67193	20.55830	20.50785	27.52600	27.61978	33.95840	34.00771
1400.0	13.67500	13.65015	22.17170	22.11449	29.71200	29.79682	36.67090	36.69713
1600.0	14.57000	14.56063	23.67340	23.60984	31.74670	31.82305	39.19580	39.20025
1800.0	15.40960	15.41575	25.03700	25.01426	33.65780	33.72605	41.56730	41.55113
2000.0	16.20400	16.22452	26.41770	26.34255	35.46540	35.52591	43.81030	43.77458

Table 2 g_w' as function of g_w and β

	$g_w = 0.2$		$g_w = 0.4$		$g_w = 0.6$		$g_w = 0.8$	
β	Approx.	Exact	Approx.	Exact	Approx.	Exact	Approx.	Exact
0.0	0.37568	0.37600	0.28176	0.28200	0.18784	0.18800	0.09392	0.09400
0.5	0.40359	0.42373	0.30839	0.32544	0.20913	0.22153	0.10623	0.11293
1.0	0.41755	0.43614	0.32140	0.33673	0.21927	0.23025	0.11195	0.11785
1.5	0.42668	0.44344	0.32987	0.34337	0.22583	0.23538	0.11562	0.12075
2.0	0.43337	0.44856	0.33607	0.34803	0.23063	0.23898	0.11830	0.12278
10.0	0.46930	0.47504	0.36941	0.37213	0.25622	0.25758	0.13249	0.13328
20.0	0.48211	0.48481	0.38133	0.38102	0.26531	0.26444	0.13749	0.13716
50.0	0.49573	0.49599	0.39398	0.39120	0.27491	0.27230	0.14274	0.14159
100.0	0.50363	0.50317	0.40131	0.39773	0.28041	0.27734	0.14574	0.14444
200.0	0.50975	0.50931	0.40694	0.40332	0.28464	0.28165	0.14803	0.14688
300.0	0.51262	0.51246	0.40957	0.40619	0.28660	0.28387	0.14909	0.14813
400.0	0.51439	0.51451	0.41118	0.41805	0.28780	0.28531	0.14974	0.14894
500.0	0.51561	0.51601	0.41230	0.40941	0.28863	0.28636	0.15018	0.14953
600.0	0.51653	0.51717	0.41313	0.41047	0.28925	0.28718	0.15051	0.14999
700.0	0.51726	0.51811	0.41379	0.41132	0.28974	0.28784	0.15078	0.15037
800.0	0.51784	0.51889	0.41432	0.41204	0.29013	0.28839	0.15099	0.15068
900.0	0.51833	0.51956	0.41479	0.41265	0.29046	0.28886	0.15116	0.15094
1000.0	0.51875	0.52014	0.41514	0.41318	0.29074	0.28927	0.15132	0.15117
1200.0	0.51943	0.52111	0.41575	0.41406	0.29119	0.28995	0.15166	0.15156
1400.0	0.51996	0.52190	0.41623	0.41477	0.29154	0.29050	0.15175	0.15187
1600.0	0.52040	0.52255	0.41662	0.41537	0.29183	0.29096	0.15190	0.15213
1800.0	0.52075	0.52311	0.41694	0.41587	0.29206	0.29135	0.15203	0.15235
2000.0	0.52105	0.52359	0.41721	0.41631	0.29227	0.29169	0.15214	0.15254

sionless distance σ is related to the similarity lateral coordinate η by $\sigma = \eta/\eta_e$.

This velocity profile appears to work very satisfactorily for adiabatic boundary layers when the pressure gradient is low or Λ is around 6 or less. When Λ is larger than 12, the velocity profiles are not altogether realistic. We propose to modify this expression by making the power of $(1-\sigma)$ larger than three. When this is done, the whole expression must be modified, resulting in the formula

$$\frac{u}{u_e} = 1 - 0.5(1-\sigma)^n[2+(n-1)\sigma] + \frac{\Lambda}{2n}\sigma(1-\sigma)^n$$

and

$$n = 1.3\exp(-10\beta g_w) + 0.5(1+4\Lambda)^{1/2}$$

When the flow is not adiabatic, the profile must be multiplied by another term related to the wall temperature. Also, $u/u_e = q(u/u_e)_{\mathrm{ad}}$. A suitable term is

$$q[(g_w^{1/2} + 2(1 - g_w^{1/2}\sigma^{0.66} + (1-g_w^{1/2})\sigma^{1.9}]^{\alpha}$$

where α is given by

$$\alpha = 1 - \exp[-(1.75g_w - 0.05)\beta^{0.2+g_w}]$$

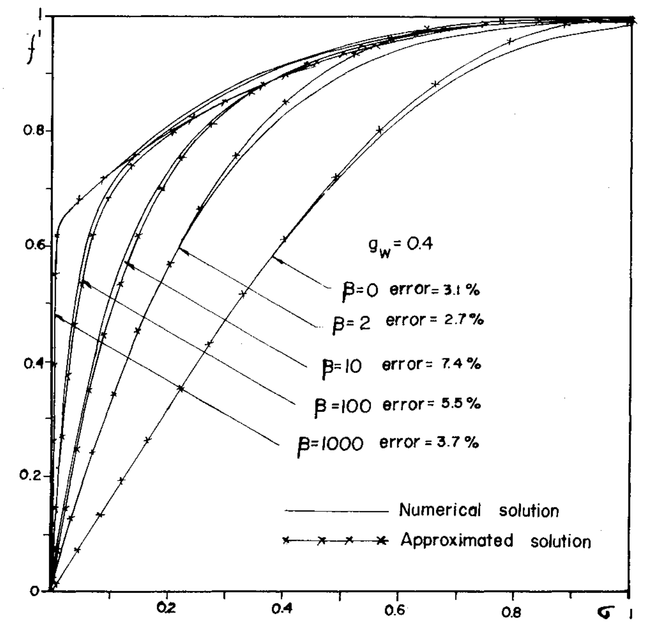

Fig. 7 Dimensionless velocity profile f' for various pressure gradient parameter β ($g_w = 0.4$).

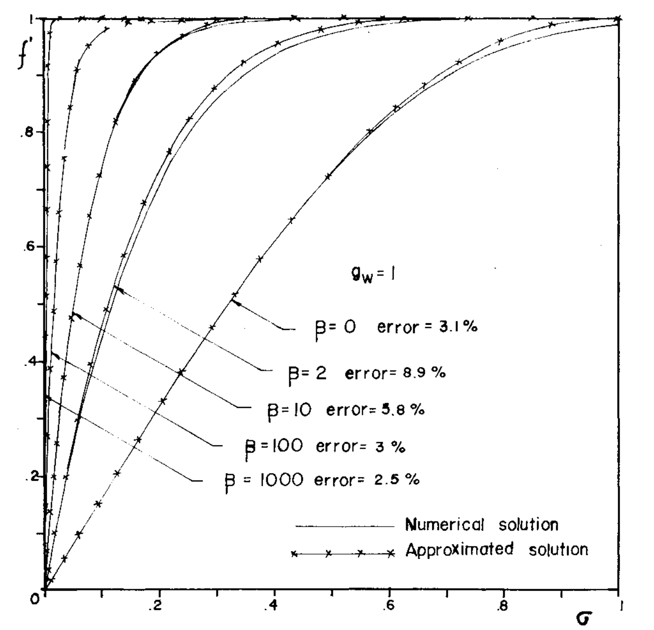

Fig. 8 Dimensionless velocity profiles f' for various pressure gradient parameter β ($g_w = 1$).

This velocity profile is compared with the exact one (as calculated numerically) in Figs. 7 and 8 for the cases of $g_w = 0.4$ and 1, respectively. Good agreement is obtained and the maximum error is less than 7%.

The Temperature Profile

The temperature profile is often calculated as a universal function of the velocity f'. For the adiabatic zero pressure gradient case, this may be done analytically to yield the Crocco integral. In the present terminology, this relation may be written as

$$g = g_w + (g_w'/f_w'')f'$$

On the other hand, we have seen that for a large pressure gradient parameter β we have

$$\lim_{\beta \to \infty} (g - f'^2) = 0, \qquad \eta \neq 0$$

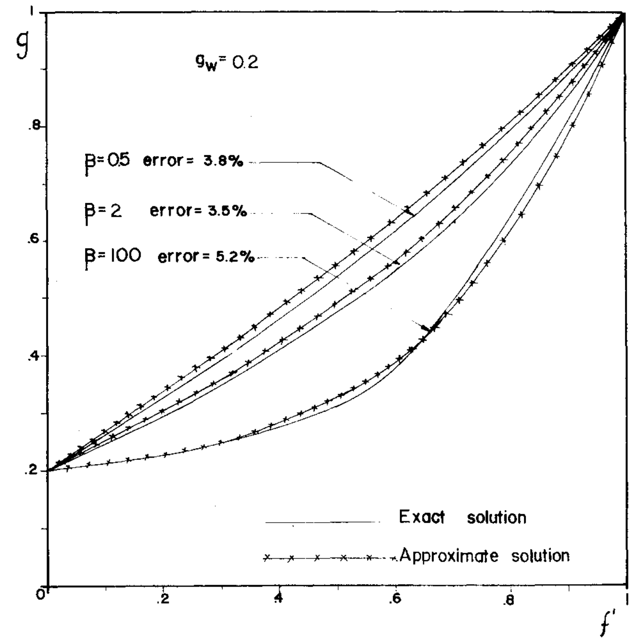

Fig. 9 $g(f')$ profile for various pressure gradient parameter β ($g_w = 0.2$).

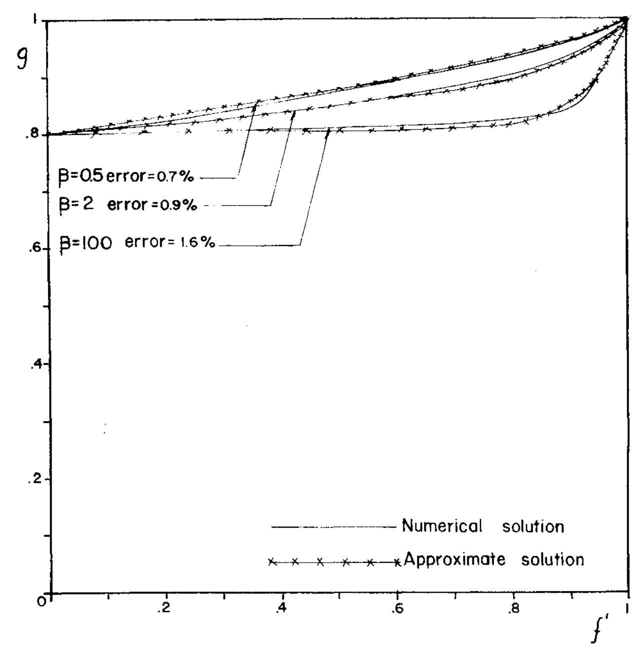

Fig. 10 $g(f')$ profile for various pressure gradient parameter β ($g_w = 0.8$).

On the basis of these two limits and using our numerical calculations, we propose the following extension of the Crocco integral:

$$g = a_0 + a_1 f' + a_2 f'^2 + a_3 f'^3 + a_4 f'^{\alpha}$$

with

$$a_0 = g_w, \quad a_1 = g_w'/f_w'' \quad a_2 = g_w g_w' \beta / (2 f_w''^3)$$

$$a_3 = 1 - (a_0 + a_1 + a_2 + a_4)$$

$$a_4 = (g_1 - 3 + 3a_0 + 2a_1 + a_2)/(\alpha - 3)$$

$$g_1 = 2.56 + 0.1 g_w - (1.56 - 1.1 g_w)/(1 + 0.3\beta)$$

$$\alpha = g_w (32.9167 - 92.5 g_w + 89.5833 g_w^2)$$

It should be noted that while the constants a_0, a_1, and a_2 are obtained so as to satisfy the governing differential equations exactly, the constants a_3 and a_4 are empirical and chosen so as to yield a good fit with the numerical solutions. These temperature profiles appear to be quite satisfactory, as may be seen in Figs. 9 and 10 showing g vs f' for various values of β and the two previous values of g_w of 0.4 and 1.

Discussion and Conclusion

This paper is concerned with highly accelerated flows. In such cases, the mainstream velocity skin friction increases very rapidly, while the boundary-layer thickness decreases quite sharply. We have proved the existence of a unique solution to the coupled momentum and energy equations governing compressible accelerating flow in laminar plane boundary layers over isothermal walls. The skin friction grows with the pressure gradient as $\beta^{1/2}$, resulting in extreme frictional losses. However, the heat transfer is bounded and the Reynolds analogy clearly does not hold in such cases. Both the velocity and temperature profiles become steeper with increasing pressure gradient and the dimensionless stagnation temperature g tends toward the velocity square f'^2 everywhere apart from the wall. It was suggested in the introduction that the difference between the similar and non-similar profiles tends to vanish under the condition of a very large pressure gradient (e.g., Fig. 1).

Altogether, the influence of the pressure gradient parameter β is of importance only until $\beta \approx 100$, especially when the temperature profiles are included. Further, a limiting profile is approached for both the velocity and the temperature.

Although the Reynolds analogy and the Crocco integral relation between velocity and temperature do not hold in these circumstances, we were able to find semiempirical formulas for the skin friction, wall heat flux, and velocity and temperature profiles. All of these relations are fairly accurate, the error variation in most cases being no more than 5%.

Another important conclusion of this work is that it is not always necessary to solve the boundary-layer equations to obtain a reliable prediction of the important parameters. The correlations suggested in this paper appear to give quite satisfactory results and they are very easy and inexpensive to implement, thus allowing a fast and economical design tool.

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